Variational and Topological methods : Theory, Applications, Numerical Simulations, and Open Problems

6-9 June 2012, Northern Arizona University

Some methods using monotonicity for solving quasilinear parabolic equations

(an introductory lecture for graduate students and postdocs)

Jacques Giacomoni
LMAP (UMR 5142), Université de Pau et des pays de l'Adour
Avenue de l'université, 64013 Pau Cedex.
(jgiacomo@univ-pau.fr)

We are first concerned by the following problem :

$$(\mathrm{P}_t)\begin{cases} u_t - \Delta_p u = f(x,u) \text{ in } Q_T \stackrel{\text{def}}{=} (0,T)\times\Omega \\ u = 0 \text{ on } \Sigma_T = [0,T]\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot) \text{ in } \Omega \end{cases}$$

i) Ω is a bounded domain in $\boldsymbol{R}^d$ with smooth boundary,

ii) $1 < p < \infty$, $\Delta_p u \stackrel{\text{def}}{=} \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, $T > 0$, $u_0 \in L^\infty(\Omega)\,(\cap W_0^{1,p}(\Omega))$.

iii) $f\ :\ (x,s)\in\Omega\times\boldsymbol{R}\to f(x,s)$ Caratheodory function, locally Lipschitz with respect to s uniformly in $x\in\Omega$ and satisfying the growth condition :

$$|f(x,s)| \le a|s|^q + b \quad \text{a.a. } x\in\Omega \tag{1}$$

with $q > 0$, $a > 0$and $b \ge 0$.

Issues :

1. local existence of weak solutions,
2. regularity of weak solutions.

Such problems arise in different models :

1) non Newtonian flows (pseudo-plastic fluids : $1 < p < 2$, dilatant fluids : $p > 2$),
2) Models of flow through porous media in turbulent regimes (ex. flow through rock filled dams [Díaz-De Thelin]),
3) models in glaceology, reaction diffusion problems, petroleum extraction, climate models, etc.

Bibliography :

- I.J. Díaz, *Nonlinear partial differential equations and free boundaries*, Elliptic Equations, Research Notes in Mathematics, Vol. I, 106, Pitman London, 1985.
- R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Volume I and II. Clarenton Press, Oxford, 1975.

Auxilary problem :

$$(\mathrm{S}_t)\begin{cases} u_t - \Delta_p u = h(x,t) \ \text{ in } \ Q_T \stackrel{\text{def}}{=} (0,T)\times\Omega \\ u = 0 \ \text{ on } \Sigma_T = [0,T]\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot) \ \text{ in } \Omega \end{cases}$$

with $T > 0$, $q \geq \min(2, p')$, $u_0 \in L^q(\Omega)$, $h \in L^1(0,T; L^q(\Omega))$.

Properties of p-Laplace operator $(1 < p < \infty)$:

- $W_0^{1,p}(\Omega) \ni u \to -\Delta_p u \in W^{-1,p'}(\Omega)$ isomorphim ;
- monotonicity :

$$u,\ v \in W_0^{1,p}(\Omega) \to < -\Delta_p u + \Delta_p v, u - v > \geq 0;$$

  aplications : weak comparison principle, Minty-Browder theory.
- variational : $-\Delta_p u = J'(u)$ with $J(v) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x$, $v \in W_0^{1,p}(\Omega)$; J convex, s.c.i. $\to -\Delta_p$ maximal monotone in $L^2(\Omega)$.

Maximal monotone operators, m-accretive operators

Let H a Hilbert space and $A : \mathcal{D}(A) \subset H \to \mathcal{P}(H)$ (multi-valued)

Definition

A is maximal monotone in H if

1. A is monotone : $\forall u, v \in \mathcal{D}(A)$, $\forall \psi \in Au$, $\forall \eta \in Av$, $(\psi - \eta | u - v) \geq 0$ ($(\cdot|\cdot)$ denotes the scalar product).
2. A is maximal : $R(I + A) = H$.

Remarks :

1. A is maximal monotone $\to J_\lambda \overset{\mathrm{def}}{=} (I + \lambda A)^{-1}$ is a contraction in H for any $\lambda > 0$.
2. A single valued, linear A maximal monotone $\Rightarrow \mathcal{D}(A)$ dense in H.

Example : Let V a reflexive Banach space such that $V \hookrightarrow H \hookrightarrow V'$ (densely and continuously) and $A : V \to V'$ single-valued, hemicontinuous, coercive and monotone. Then $A|_H$ is maximal monotone (by Minty-Browder Theorem).

Let X a banach space. $A \subset X \times X$.

Definition

A is m-accretive in X if

1. (accretive) $\forall [x_1, y_1], [x_2, y_2] \in A$ and $\lambda > 0$, $\|x_1 - x_2\|_X \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|_X$.
2. (maximal) $R(I + A) = X$.

Example 1 :
$-\Delta_p$ is m-accretive in $L^q(\Omega)$ for $q \geq \min(2, \frac{Np}{Np-N+p})$ (accretive for $q \geq 1$). Indeed, introduce for $\lambda > 0$ and $f \in L^q(\Omega)$ the energy functional E defined by

$$E(u) \stackrel{\text{def}}{=} \frac{1}{2} \int_\Omega u^2 \, \mathrm{d}x + \frac{\lambda}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega fu \, \mathrm{d}x.$$

E is convex and s.c.i. in $L^q(\Omega) \cap W_0^{1,p}(\Omega)$ for $q \geq \min(2, \frac{Np}{Np-N+p})$.

Remark : By the weak comparison principle, $-\Delta_p$ is m-accretive in $L^\infty(\Omega)$.

Example 2 :

Let β a maximal monotone operator in $\boldsymbol{R}$, $q \geq 1$. Let $\tilde{\beta}$ the realization of β in $L^q(\Omega)$. Then, A defined by $Au \stackrel{\text{def}}{=} -\Delta u + \tilde{\beta}(u)$, is m-accretive in $L^q(\Omega)$.

Other examples : porous media equation involves $-\Delta(\beta(u))$ [with $0 \in \beta(0)$ and β maximal monotone in $\boldsymbol{R}$] which is m-accretive in $L^1(\Omega)$.

Theory of maximal monotone or m-accretive operators $\rightarrow$ existence of weak solutions to

$$\frac{du}{dt} + Au \ni f, \quad u(0) = u_0. \tag{2}$$

References : H. Brezis *Opérateurs maximaux monotones et semi-groupes de contractions dans des espaces de Hilbert*,
V. Barbu, *Non-Linear Differential Equations of Monotone Type On Banach Space*.

Definition

Let $\epsilon > 0$, $T > 0$, $N \in \mathbb{N}\backslash\{0\}$ and $f \in L^1(0, T; X)$.

1. An ϵ-*discretization* on $[0, T]$ (denoted by $D^\epsilon(t_1, \cdots, t_{N-1}; f_1, \cdots, f_N)$) of equation in (2) consists of a partition $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N = T$ and a finite sequence $\{f_i\}_{i=1}^N$ such that

$$t_i - t_{i-1} < \epsilon \text{ for } i = 1, \cdots, N \text{ and } \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| \,\mathrm{d}s < \epsilon.$$

2. An ϵ-approximate solution to (2) associated to $D^\epsilon(t_1, \cdots, t_{N-1}; f_1, \cdots, f_N)$ is a piecewise constant function $z : [0, T] \to X$ whose (nodal) values z_i satisfies on $(t_{i-1}, t_i]$ satisfy

$$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + Az_i \ni f_i, \quad i = 1, \cdots, N \text{ and } \|z(0) - u_0\| \leq \epsilon.$$

Definition

Let A be m-accretive in X.

1. A *mild* solution to the Cauchy problem (2) is a function $y \in C([0, T]; X)$ such that for any $\epsilon > 0$, there is ϵ-approximate solution z on $[0, T]$ such that $\sup_{t\in[0,T]} \|y(t) - z(t)\| \le \epsilon$ and $y(0) = u_0$.
2. A *strong* solution to the Cauchy problem (2) is a function $y \in W^{1,1}((0, T]; X) \cap C([0, T]; X)$ such that

$$f(t) - \frac{dy}{dt}(t) \in Ay(t), \ \text{ a.e. } t \in (0, T), \quad y(0) = u_0.$$

Remark. A strong solution to (2) is unique and continuous in respect to f and u_0.

Theorem. (existence of mild solutions)

Let A be m-accretive in X and $u_0 \in \overline{\mathcal{D}(A)}^X$. Then, there exists a unique mild solution to (2).

Proof. (main steps)

Let $r, k \in L^1(0,T;X)$, $\epsilon > 0$, $\eta > 0$ and u_ϵ, v_η be ϵ-approximate solution and η-approximate solution respectively associated to the discretisations $D^\epsilon(i\epsilon, r^i \overset{\text{def}}{=} \frac{1}{\epsilon}\int_{(i-1)\epsilon}^{i\epsilon} r(s)\,\mathrm{d}s)$ and $D^\eta(i\eta, k^i \overset{\text{def}}{=} \frac{1}{\eta}\int_{(i-1)\eta}^{i\eta} k(s)\,\mathrm{d}s)$ with initial data u_0 and v_0 in $\overline{\mathcal{D}(A)}^X$.

We define

$$\varphi(t,s) \overset{\text{def}}{=} \|r(t) - k(s)\|_X \quad (t,s) \in [0,T] \times [0,T]$$

and for a fixed $z \in \mathcal{D}(A)$

$$\begin{aligned} b(t,r,k) \quad &\overset{\text{def}}{=} \quad \|u_0 - z\|_X + \|v_0 - z\|_X + |t|\|Az\|_X \\ &+ \quad \int_0^{t^+} \|r(\tau)\|_X \mathrm{d}\tau + \int_0^{t^-} \|k(\tau)\|_X \mathrm{d}\tau, \ t \in [-T,T], \end{aligned}$$

and

$$\Psi(t,s) \stackrel{\text{def}}{=} b(t-s,r,k) + \begin{cases} \int_0^s \varphi(t-s+\tau,\tau)\mathrm{d}\tau & \text{if } 0 \leq s \leq t \leq T, \\ \int_0^t \varphi(\tau, s-t+\tau)\mathrm{d}\tau & \text{if } 0 \leq t \leq s \leq T, \end{cases}$$

the solution to the transport equation

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t,s) + \frac{\partial \Psi}{\partial s}(t,s) &=& \varphi(t,s) & (t,s) \in [0,T] \times [0,T], \\ \Psi(t,0) &=& b(t,r,k) & t \in [0,T], \\ \Psi(0,s) &=& b(-s,r,k) & s \in [0,T]. \end{cases} \tag{3}$$

Finally, Let $\{u_\epsilon^n\}$ and $\{v_\eta^n\}$ be the nodal functions associated to u_ϵ and v_η respectively. For $n, m \in \mathbb{N}^*$,

$$\begin{aligned} u_\epsilon^n - v_\eta^m + \frac{\epsilon\eta}{\epsilon+\eta}(Au_\epsilon^n - Av_\eta^m) &= \frac{\eta}{\epsilon+\eta}(u_\epsilon^{n-1} - v_\eta^m) \\ + \frac{\epsilon}{\epsilon+\eta}(u_\epsilon^n - v_\eta^{m-1}) &+ \frac{\epsilon\eta}{\epsilon+\eta}(r^n - k^m), \end{aligned}$$

Accretivity of A in X implies that $\Phi^{\epsilon,\eta}_{n,m} = \|u^n_\epsilon - v^m_\eta\|_X$ satisfies

$$\begin{aligned}
\Phi^{\epsilon,\eta}_{n,m} &\leq \frac{\eta}{\epsilon+\eta}\Phi^{\epsilon,\eta}_{n-1,m} + \frac{\epsilon}{\epsilon+\eta}\Phi^{\epsilon,\eta}_{n,m-1} + \frac{\epsilon\eta}{\epsilon+\eta}\|r^n - k^m\|_X, \\
\Phi^{\epsilon,\eta}_{n,0} &\leq b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Phi^{\epsilon,\eta}_{0,m} \leq b(-s_m, r_\epsilon, k_\eta),
\end{aligned}$$

For the two last inequalities, note that from accretivity of A, we have :

$$\begin{aligned}
\|u^n_\epsilon - z\|_X &\leq \|u^n_\epsilon - z + \epsilon(r^n + \frac{1}{\epsilon}(-u^n_\epsilon + u^{n-1}_\epsilon) - Az)\|_X \\
&\leq \|u^{n-1}_\epsilon - z\|_X + \epsilon\|r^n\|_X + \epsilon\|Az\|_X,
\end{aligned}$$

$$\begin{aligned}
\|v^m_\eta - z\|_X &\leq \|v^m_\eta - z + \eta(r^m + \frac{1}{\eta}(-v^m_\eta + v^{m-1}_\eta) - Az)\|_X \\
&\leq \|v^{m-1}_\eta - z\|_X + \eta\|k^m\|_X + \eta\|Az\|_X,
\end{aligned}$$

Clearly, $\Phi^{\epsilon,\eta}_{n,m} \le \Psi^{\epsilon,\eta}_{n,m}$ where $\Psi^{\epsilon,\eta}_{n,m}$ satisfies

$$\begin{aligned}
\Psi^{\epsilon,\eta}_{n,m} &= \frac{\eta}{\epsilon+\eta}\Psi^{\epsilon,\eta}_{n-1,m} + \frac{\epsilon}{\epsilon+\eta}\Psi^{\epsilon,\eta}_{n,m-1} + \frac{\epsilon\eta}{\epsilon+\eta}\|r^n - k^m\|_X, \\
\Psi^{\epsilon,\eta}_{n,0} &= b(t_n, r_\epsilon, k_\eta) \quad \text{and} \quad \Psi^{\epsilon,\eta}_{0,m} = b(-s_m, r_\epsilon, k_\eta).
\end{aligned}$$

We have

$$b(\cdot, r_\epsilon, k_\eta) \to b(\cdot, r, k) \text{ in } L^\infty(-T, T; X) \quad \text{and}$$

$$\Phi^{\epsilon,\eta} \to \varphi \text{ in } L^1((0,T)\times(0,T); X).$$

Then,

$$\rho_{\epsilon,\eta} = \|\Psi^{\epsilon,\eta} - \Psi\|_{L^\infty([0,T]\times[0,T])} \to 0 \text{ as } (\epsilon,\eta) \to 0.$$

We get for $t \in [0,T]$ and $s \in [0,T]$

$$\|u_\epsilon(t) - v_\eta(s)\|_X = \Phi^{\epsilon,\eta}(t,s) \le \Psi^{\epsilon,\eta}(t,s) \le \Psi(t,s) + \rho_{\epsilon,\eta}, \qquad (4)$$

Doing in (4) $t=s$, $r=k$, $v_0=u_0$:

$$\|u_\epsilon(t)-u_\eta(t)\|_X \le 2\|u_0-z\|_X+\rho_{\epsilon,\eta}.$$

We have also

$$\begin{aligned}
\|u_\epsilon(t)-\tilde{u}_\epsilon(t)\|_X &\le 2\|u_\epsilon(t)-u_\epsilon(t-\epsilon)\|_X \\
&\le 4\|u_0-z\|_X+2\epsilon\|Az\|_X+2\int_0^\epsilon \|h(\tau)\|_X \mathrm{d}\tau \\
&+ 2\int_0^t \|h(\epsilon+\tau)-h(\tau)\|_X \mathrm{d}\tau.
\end{aligned}$$

From the fact that $u_0\in\overline{\mathcal{D}(A)}^X$, $\{u_\epsilon\}_{\epsilon>0}$, $\{\tilde{u}_\epsilon\}_{\epsilon>0}$ are Cauchy sequences in $L^\infty(0,T;X)$ which converge to $u\in C([0,T];X)$ which satisfies $u(0)=u_0$ and (by (4) with $v_0=u_0$)

$$\begin{aligned}
\|u(t)-u(s)\|_X &\le 2\|u_0-z\|_X+|t-s|\|Az\|_\infty+\int_0^{|t-s|} \|h(\tau)\|_X \mathrm{d}\tau \\
&+ \int_0^{\max(t,s)} \|h(|t-s|+\tau)-h(\tau)\|_X \mathrm{d}\tau. \qquad (5)
\end{aligned}$$

Passing to the limit in (4) with $t = s$ we obtain

$$\|u(t) - v(t)\|_X \leq \|u_0 - z\|_X + \|v_0 - z\|_X + \int_0^t \|r(\tau) - k(\tau)\|_X \mathrm{d}\tau,$$

Since $v_0 \in \overline{\mathcal{D}(A)}^X$, we get

$$\|u(t)-v(t)\|_X \leq \|u_0 - v_0\|_X + \int_0^t \|r(s) - k(s)\|_X \mathrm{d}s, \quad 0 \leq t \leq T. \tag{6}$$

which implies that u is continuous in respect to u_0 and f.
If $u_0 \in \mathcal{D}(A)$ and $h \in W^{1,1}(0, T; X)$, then u is absolutely continuous. Indeed (for $s \leq t$),

$$\begin{aligned} \|u(t) - u(s)\|_X &\leq \|u_0 - u(t-s)\|_X + \int_0^s \|h(\tau) - h(\tau + t - s)\|_X \mathrm{d}\tau \\ &\leq (t-s)\left((\|Au_0 - h(0)\|_X + \int_0^T \left\|\frac{\mathrm{d}h(\tau)}{\mathrm{d}t}\right\|_X \mathrm{d}\tau\right). \end{aligned}$$

Remark. A generates a continuous semi-group of contractions on $\overline{\mathcal{D}(A)}^X$, that is a family of mappings that maps $\overline{\mathcal{D}(A)}^X$ into itself with the properties :

(i) $S(t+s)x = S(t)S(s)x$, $\forall x \in \overline{\mathcal{D}(A)}^X$, $t, s \geq 0$.

(ii) $S(0)x = x$, $\forall x \in \overline{\mathcal{D}(A)}^X$.

(iii) For every $x \in \overline{\mathcal{D}(A)}^X$, the function $t \to S(t)x$ is continuous on $[0, \infty)$ and solves (2) with $f = 0$.

(iv) $\|S(t)x - S(t)y\|_X \leq \|x - y\|_X$, $\forall t \geq 0$, $x, y \in \overline{\mathcal{D}(A)}^X$.

Aplications : Δ_p is the infinitesimal generator of a continuous semi-group of contractions in $L^q(\Omega)$ with $q \in [2, \infty)$ and in $C_0(\overline{\Omega})$ $(q = \infty)$ with

$$\mathcal{D}(A) = \left\{ u \in L^q(\Omega) \cap W_0^{1,p}(\Omega) | -\Delta_p u \in L^q(\Omega) \right\}, \; q \in [2, \infty].$$

Concerning (P_t), (local) existence of weak solutions can be proved as follows :

1. Existence (by Schauder fixed-point theorem) and uniqueness of global solutions $u_k \in L^\infty(0, T; W_0^{1,p}(\Omega) \cap L^\infty(Q)$ such that $\frac{\partial u_k}{\partial t} \in L^2(Q)$ to

$$(\mathrm{Pk}_t) \begin{cases} u_t - \Delta_p u = f_k(x, u) \text{ in } Q \stackrel{\text{def}}{=} (0, T) \times \Omega \\ u = 0 \text{ on } \Sigma_T = [0, T] \times \partial\Omega, \; u > 0 \text{ in } Q \\ u(0, \cdot) = u_0(\cdot) \text{ in } \Omega \end{cases}$$

   where

$$f_k(x, t) \stackrel{\text{def}}{=} f(x, T_k(t)) \text{ with } T_k(t) \stackrel{\text{def}}{=} \begin{cases} \mathrm{sign}(t)k \text{ if } |t| \geq k \\ t \text{ if } -k \leq t \leq k \end{cases}$$

2. Existence of barrier functions or a priori estimates $\rightarrow u_k$ weak solution to $(\mathrm{P_t})$ for $T > 0$ small enough.
3. compact semi-group, maximal time interval $[0, T^*)$: blow up in finite time if $T^* < \infty$ (see [Vrabie]).

We can adapt the method of semi-discretization in time on the following quasilinear Barenblatt equation :

$$(\mathrm{P})\begin{cases} f(\cdot,\partial_t u) - \Delta_{\tilde{\mathbf{p}}} u = g \ \text{ in } \ Q =]0,T[\times\Omega, \\ u = 0 \ \text{ on } \Sigma = (0,T)\times\Gamma, \quad u(x,0) = u_0(x) \ \text{ in } \Omega \end{cases}$$

where $\Delta_{\tilde{\mathbf{p}}} u \stackrel{\text{def}}{=} \sum_{i=1}^{d} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\,\partial_{x_i} u\right)$ is the anisotropic $\tilde{\mathbf{p}}(x)$-Laplace operator such that $\tilde{\mathbf{p}} = (p_i)_{i=1,..,d} : \Omega \mapsto [p^-, p^+]$ with $\max(1, \frac{2d}{d+2}) < p^- \le p^+ < \infty$, continuous in Ω with logarithmic module of continuity.

f satisfies :

(f1) $f : (x,t) \in \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is a Carathéodory function, non-decreasing with respect to t and

(f2) There exist $a_1 \in L^2(\Omega)$ and $a_2 \ge 0$ such that

$$x \in \Omega \ a.e., \ \ t \in \boldsymbol{R}, \ \ |f(x,t)| \le a_1(x) + a_2|t|, \tag{7}$$

(f3) there exist $b_1 \in L^1(\Omega)$ and $b_2 \ge 0$ such that,

$$x \in \Omega \ a.e., \ \ t \in \boldsymbol{R}, \ \ tf(x,t) \ge b_2|t|^2 - b_1(x). \tag{8}$$

Concerning g and the initial data u_0, we require

(g1) $g \in L^2(Q)$ and $u_0 \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$.

Basic properties of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$

Let h a measurable function. Set $\rho_p(h) \stackrel{\text{def}}{=} \int_\Omega |h(x)|^{p(x)} \mathrm{d}x$ and the luxembourg norm

$$\|f\|_p \stackrel{\text{def}}{=} \inf \left\{ \lambda > 0,\ \rho_p\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

We define

$$\begin{aligned} L^{p(x)}(\Omega) &= \left\{ u : \Omega \to \boldsymbol{R},\ \text{measurable}/\ x \mapsto |u(x)|^{p(x)} \in L^1(\Omega) \right\}, \\ W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega) &= \left\{ u \in W_0^{1,1}(\Omega)\ /\ \partial_{x_i} u \in L^{p_i(x)}(\Omega),\ i = 1, .., d \right\}. \end{aligned}$$

1. $(L^{p(x)}(\Omega), \|f\|_p)$ is a separable uniform convex space.
2. Dual space : $L^{p'(x)}(\Omega)$ where $p'(x) = \frac{p(x)}{p(x)-1}$ and Hölder inequality :

$$\int_\Omega |f(x)g(x)| \, \mathrm{d}x \leq c\|f\|_p\|g\|_{p'}. \tag{9}$$

3. Let $(u_n)_{n\geq 1} \subset L^{p(x)}(\Omega)$. Then, $\lim\limits_{n\to\infty} u_n = 0$ in $L^{p(x)}(\Omega)$ if and only if $\lim_n \rho_p(u_n) = 0$.
   If $u_n \rightharpoonup u$ in $L^{p(x)}(\Omega)$, then
   $\displaystyle\int_\Omega |u(x)|^{p(x)}\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega |u_n(x)|^{p(x)}\mathrm{d}x.$
4. $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ if and only if $q(x) \leq p(x)$.
5. Endowed with the graph norm $\|u\|_{W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)} = \sum_{i=1}^d \|\partial_{x_i} u\|_{p_i}$, $W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$ is a reflexive and separable Banach space.

Morawetz Inequality

Lemma

Let $1 < p < \infty$*. Set* $s \stackrel{\text{def}}{=} \min(1, \frac{p}{2})$ *and*

$$C(p) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{1-2^{1-p}}\right)^{\frac{p}{2}} \text{ if } p < 2 \\ \frac{1}{2} \text{ if } p \geq 2. \end{cases}$$

Then, $\forall\, a,\ b \in \boldsymbol{R}$,

$$\left|\frac{a-b}{2}\right|^p \leq C(p)\left(|a|^p + |b|^p\right)^{1-s}\left(|a|^p + |b|^p - 2\left|\frac{a+b}{2}\right|^p\right)^s. \tag{10}$$

Compactness

If $p_i(x) \leq q_i(x)$ for any i, then $W_0^{1,\tilde{\mathbf{q}}(x)}(\Omega) \hookrightarrow W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$.
Then, if $p^- > \frac{2d}{d+2}$, $W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega) \hookrightarrow\!\!\!\!\rightarrow L^2(\Omega)$.

Definition

A solution of (P) is any function $u \in L^\infty(0,T; W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$ with $\partial_t u \in L^2(Q)$ such that $u(0,.) = u_0$ and, for any $v \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$, a.e. in $]0,T[$,

$$\int_\Omega f(.,\partial_t u) v \mathrm{d}x + a_{\tilde{\mathbf{p}}}(u,v) = \int_\Omega g v \, \mathrm{d}x.$$

We show :

Theorem

(G, Vallet) *Assume that* $u_0 \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$ *and the conditions (f1)-(f3) and (g1). Then, there exists a solution* u *to* (P). *Furthermore,* $u \in C([0,T[, W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$.

Semi-discretization in time : $N \in \boldsymbol{N}\backslash\{0\}$, $\Delta_t \stackrel{\text{def}}{=} \frac{T}{N}$,

$$\left\{ \begin{array}{c} f(., \frac{u^n - u^{n-1}}{\Delta_t}) - \Delta_{\tilde{\mathbf{p}}} u^n = g^n \quad \text{in } \Omega \\ u^n|_{\partial\Omega} = 0 \end{array} \right.$$

with $g^n \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{(n-1)\Delta_t}^{n\Delta_t} g(s)\mathrm{d}\, s \quad$ on $[(n-1)\Delta_t, n\Delta_t)$.
$u^0 = u_0$ and
$u^n \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$ is the unique global minimizer to E defined by

$$E(u) = \Delta_t \int_\Omega F(\cdot, \frac{u - u^{n-1}}{\Delta_t})\,\mathrm{d}x + \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\,\mathrm{d}x - \int_\Omega g^n u\,\mathrm{d}x$$

where $F(., \lambda) = \int_0^\lambda f(., \sigma) d\sigma$.

We define for $t \in ((n-1)\Delta_t, n\Delta_t]$

$$g_{\Delta_t}(t) = g^n, \; u_{\Delta_t}(t) = u^n,$$
$$\tilde{u}_{\Delta_t}(t) = \frac{t-(n-1)\Delta_t}{\Delta_t}(u^n - u^{n-1}) + u^{n-1}.$$

Note that

$$\|g_{\Delta_t}\|^2_{L^2(Q)} = \Delta_t \sum_{n=1}^{N} \|g^n\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(Q)},$$

Thus,

$$f(\cdot, \partial_t \tilde{u}_{\Delta_t}) - \Delta_{\tilde{\mathbf{p}}} u_{\Delta_t} = g_{\Delta_t} \text{ in } Q_T =]0, T[\times\Omega.$$

Energy estimates $\rightarrow$

$$\|\partial_t \tilde{u}_{\Delta_t}\|_{L^2(Q)} \leq C,$$

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \text{ are bounded in } L^\infty(0, T; W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)),$$

$$\|\tilde{u}_{\Delta_t} - u_{\Delta_t}\|_{L^2(Q)} \leq \Delta_t ||\partial_t \tilde{u}_{\Delta_t}||_{L^2(Q)} \leq C\Delta_t.$$

up to a subsequence,

$$\begin{aligned}
\tilde{u}_{\Delta_t} &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0, T; W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)),\\
u_{\Delta_t} &\overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(0, T; W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)),\\
\tilde{u}_{\Delta_t}, u_{\Delta_t} &\overset{*}{\rightharpoonup} u = v \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \text{ and}\\
\partial_t \tilde{u}_{\Delta_t} &\rightharpoonup \partial_t u \quad \text{in} \quad L^2(Q).
\end{aligned}$$

Note that $u \in L^\infty(0, T; W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)) \cap H^1(0, T, L^2(\Omega))$ and then $u \in C_w([0, T], W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$ and $u(t) \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$, for any $t \in [0, T]$.

Strong convergence by

Theorem

(Aubin-Simon) Consider $p \in]1, +\infty[$, $q \in [1, +\infty]$ and V, E and F three Banach spaces such that $V \hookrightarrow\!\!\!\rightarrow E \hookrightarrow F$. Then, if A is a bounded subset of $W^{1,p}(0, T, F)$ and of $L^q(0, T, V)$, A is relatively compact in $C([0, T], F)$ and in $L^q(0, T, E)$.

$$u_{\Delta_t} \to u \quad \text{in } L^p(0,T;W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)) \quad \text{for any } \ 1 \le p < +\infty.$$

Consequently, setting $\tilde{\mathbf{q}}(x) = (q_i(x))$ with $q_i(x) \stackrel{\text{def}}{=} \frac{p_i(x)}{p_i(x)-1}$ one has

$$-\Delta_{\tilde{\mathbf{p}}} u_{\Delta_t} \to -\Delta_{\tilde{\mathbf{p}}} u \quad \text{in } L^{\frac{p}{p-1}}(0,T;\, W^{-1,\tilde{\mathbf{q}}(x)}(\Omega)) \quad \text{for any } \ 1 < p < +\infty.$$

Next, there exists $\chi \in L^2(Q)$ such that as $\Delta_t \to 0^+$ (up to a subsequence)

$$f(\partial_t \tilde{u}_{\Delta_t}) \rightharpoonup \chi \quad \text{in } L^2(0,T;\, L^2(\Omega)) \quad \text{and } \chi - \Delta_{\tilde{\mathbf{p}}} u = g \quad \text{in } \mathcal{D}'(Q).$$

From $g_{\Delta_t} \to g$ in $L^2(Q)$ as $\Delta_t \to 0^+$ and convexity arguments, we get

$$\limsup_{\Delta_t \to 0^+} \int_0^s \int_\Omega f(\partial_t \tilde{u}_{\Delta_t}) \partial_t \tilde{u}_{\Delta_t} \, \mathrm{d}x\mathrm{d}t + \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} \left|\partial_{x_i} u(s)\right|^{p_i(x)} \mathrm{d}x$$

$$\le \ \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} \left|\partial_{x_i} u_0\right|^{p_i(x)} \mathrm{d}x + \int_0^s \int_\Omega g \partial_t u \, \mathrm{d}x\mathrm{d}t.$$

we have for $0 < \Delta_t << 1$ that

$$\int_0^{s-\Delta_t}\int_\Omega \chi(t)\frac{u(t+\Delta_t)-u(t)}{\Delta_t} + \int_0^{s-\Delta_t} a_{\tilde{\mathbf{p}}}\left(u(t), \frac{u(t+\Delta_t)-u(t)}{\Delta_t}\right)$$
$$= \int_0^{s-\Delta_t}\int_\Omega g(t)\frac{u(t+\Delta_t)-u(t)}{\Delta_t}$$

where

$$a_{\tilde{\mathbf{p}}} : (u,v) \mapsto \sum_{i=1}^{d}\int_\Omega |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v \mathrm{d}x.$$

and as $\Delta_t \to 0^+$

$$\int_0^s\int_\Omega \chi(t)\partial_t u + \liminf_{\Delta_t}\sum_{i=1}^{d}\frac{1}{\Delta_t}\int_{s-\Delta_t}^{s}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} u(t)|^{p_i(x)}$$
$$\geq \liminf_{\Delta_t}\sum_{i=1}^{d}\frac{1}{\Delta_t}\int_0^{\Delta_t}\int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} u(t)|^{p_i(x)} + \int_0^s\int_\Omega g(t)\partial_t u.$$

Since $u \in C_w([0, T], W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$, one has

$$\int_0^s \int_\Omega \chi(t)\partial_t u + \liminf_{\Delta_t} \sum_{i=1}^d \frac{1}{\Delta_t} \int_{s-\Delta_t}^s \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u(t)|^{p_i(x)}$$

$$\geq \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_0|^{p_i(x)} \,\mathrm{d}x + \int_0^s \int_\Omega g(t)\partial_t u$$

$$\geq \limsup_{\Delta_t \to 0^+} \int_0^s \int_\Omega f(\partial_t \tilde{u}_{\Delta_t})\partial_t \tilde{u}_{\Delta_t} + \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u(s)|^{p_i(x)}.$$

Therefore, for any Lebesgue point s of u in $L^{p^+}(0, T, W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$, we get that

$$\int_0^s \int_\Omega \chi(t)\partial_t u + \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u(s)|^{p_i(x)}$$

$$\geq \limsup_{\Delta_t \to 0^+} \int_0^s \int_\Omega f(\partial_t \tilde{u}_{\Delta_t})\partial_t \tilde{u}_{\Delta_t} + \sum_{i=1}^d \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u(s)|^{p_i(x)}.$$

Thus,

$$\limsup_{\Delta_t \to 0^+} \int_0^s \int_\Omega f(\partial_t \tilde{u}_{\Delta_t}) \partial_t \tilde{u}_{\Delta_t} \,\mathrm{d}x\mathrm{d}t \leq \int_0^s \int_\Omega \chi \partial_t u \,\mathrm{d}x\mathrm{d}t.$$

From the monotonicity of f, we know that for any $v \in L^2(Q_s)$ where $Q_s \stackrel{\text{def}}{=}]0, s[\times\Omega$, one has

$$0 \leq \int_{Q_s} [f(v) - f(\partial_t \tilde{u}_{\Delta_t})](v - \partial_t \tilde{u}_{\Delta_t}) \,\mathrm{d}x\mathrm{d}t. \tag{11}$$

Taking the lim sup as $\Delta_t \to 0^+$ in (11), we obtain that

$$0 \leq \int_{Q_s} [f(v) - \chi](v - \partial_t u) \,\mathrm{d}x\mathrm{d}t.$$

A classical monotonicity argument (Minty's trick !) yields

$$\begin{aligned} \text{for } s \text{ a.e. in }]0, T[,\ \chi = f(\partial_t u) \quad \text{in } L^2(Q_s), \\ \text{that is, } \ \chi = f(\partial_t u) \quad \text{in } L^2(Q) \end{aligned}$$

and then $f(\partial_t u) - \Delta_{\tilde{\mathbf{p}}} u = g$.

Lemma

Assume that hypothesis (f1)-(f3) are satisfied and let $\bar{T} > 0$, $Q =]0, \bar{T}[\times\Omega$, $u_0 \in W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega)$, $g \in L^2(Q)$ *and* $u \in H^1(Q) \cap C_w([0, \bar{T}], W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$ *the solution of the* $\tilde{\mathbf{p}}$*-Laplace evolution equation :*

$$f(\cdot, \partial_t u) - \Delta_{\tilde{\mathbf{p}}} u = g \text{ in } Q, \quad \text{with } u(0, .) = u_0.$$

Then, $u \in C([0, \bar{T}[, W_0^{1,\tilde{\mathbf{p}}(x)}(\Omega))$ *and for any* $t \in [0, \bar{T}[$,

$$\begin{aligned} & \int_{]0,t[\times\Omega} f(\cdot, \partial_t u(\sigma)) u(\sigma) \,\mathrm{d}x\mathrm{d}\sigma + \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u(t,x)|^{p_i(x)} \\ = & \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_0|^{p_i(x)} \,\mathrm{d}x + \int_{]0,t[\times\Omega} g(\sigma) \partial_t u \,\mathrm{d}x\mathrm{d}\sigma. \end{aligned} \tag{12}$$